Fiber bundles

$F \xrightarrow{i} E \xrightarrow{p} B$ where all <u>fibers</u> $p^{-1}(b) \subset E$ are homeomorphic to each other
injective surjective (+ <u>more</u>...)

ex. trivial fibration $E = F \times B \underset{\text{proj.}}{\xrightarrow{p}} B$

Essentially, a fiber bundle is a "twisted product",
eg Möbius band = interval bundle over S^1.

These give rise to l.e.s. in homotopy gps via <u>homotopy lifting property</u>

Def:
- $p : E \to B$ has the <u>homotopy lifting property</u> wrt a space X if
  $\forall$ homotopy $g_t : X \to B$ $(t \in [0,1])$ and $\forall$ lift $\tilde{g}_0 : X \to E$ of g_0,
  $$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{g}_0"] \arrow[r, "g_0"'] & B \end{tikzcd}$$
  $\exists$ homotopy $\tilde{g}_t : X \to E$ lifting g_t.
- a <u>fibration</u> is a map $p : E \to B$ st. homotopy lifting property holds wrt all spaces X.

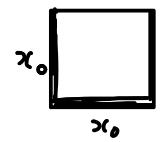
Ex: for $B \times F \xrightarrow{\text{proj.}} B$, given g_t and $\tilde{g}_0(x) = (g_0(x), h_0(x))$, take $\tilde{g}_t(x) = (g_t(x), h_0(x))$.

Thm: Suppose $p : E \to B$ has homotopy lifting property wrt disks D^k $\forall k \geq 0$.
Choose base pts $b_0 \in B$, $x_0 \in F = p^{-1}(b_0)$. Then $p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0)$ is an isom. $\forall n \geq 1$, and if B is path-connected we have
$$\cdots \to \pi_n(F, x_0) \xrightarrow{i_*} \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots \to \pi_0(E, x_0) \to 0$$

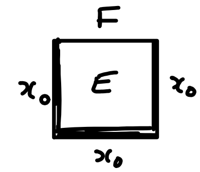
Rmk:
- in fact the proof will use a relative version for $(D^k, \partial D^k)$.
  homotopy lifting prop. for X: given map $X \times I \to B$
  and lift to E of its restr. to $X \times \{0\}$, $\exists$ lift over $X \times I$.
  For (X, A): —— " —— $(X \times \{0\}) \cup (A \times I)$, —— " ——

  Since $(D^k \times I, D^k \times 0)$ is homotopy eq. to $(D^k \times I, D^k \times 0 \cup \partial D^k \times I)$,
  homotopy lifting for $D^k \iff$ for $(D^k, \partial D^k)$.
  and in fact, by induction on cells, $\iff$ for all CW-pairs (X, A)
  (enough to extend homotopy over one cell at a time)
- map w/ homotopy lifting property for discs =: "<u>Serre fibration</u>"
  & enough for all practical purposes.

Pf. thm: 1) show p_* is onto: given $[f] \in \pi_n(B, b_0)$ rep^d by $f : (I^n, \partial I^n) \to (B, b_0)$ ②
lift f to constant map to x_0 over $J_n = (I^{n-1} \times \{0\}) \cup (\partial I^{n-1} \times I) \subset \partial I^n$

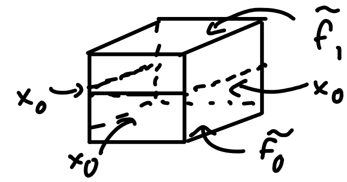


Then by homotopy ext. prop for $(I^{n-1}, \partial I^{n-1})$
this extends to a lift $\tilde{f} : I^n \to E$ of f.
Moreover, $\tilde{f}(\partial I^n) \subseteq F$ since lifts $f(\partial I^n) = b_0$.



Hence $\tilde{f}$ represents an element of $\pi_n(E, F, x_0)$
and $p_*[\tilde{f}] = [p \circ \tilde{f}] = [f]$.

2) injectivity: given $\tilde{f}_0, \tilde{f}_1 : (I^n, \partial I^n, J_n) \to (E, F, x_0)$ st.
$p_*[\tilde{f}_0] = p_*[\tilde{f}_1]$, let $G : (I^n \times I, \partial I^n \times I) \to (B, b_0)$ homotopy
from $p \circ \tilde{f}_0$ to $p \circ \tilde{f}_1$. We have a partial lift by $\tilde{f}_0$, $\tilde{f}_1$, and x_0
of G over $I^n \times \{0\} \cup I^n \times \{1\} \cup J_n \times [0,1]$



which after permuting coords. of $I^n \times I$,
is another instance of relative h. lifting problem
for $(I^n, \partial I^n)$. Hence $\exists$ lifting $\tilde{G}$ over $I^n \times I$.
By construction, on each $I^n \times \{t\}$, $\tilde{G}$ maps $J_n \times \{t\}$ to x_0
and $\partial I^n \times \{t\}$ to F since $G(\partial I^n \times \{t\}) = b_0$.
Hence $\tilde{G}$ is a homotopy of maps $(I^n, \partial I^n, J_n) \to (E, F, x_0)$
and $[\tilde{f}_0] = [\tilde{f}_1]$.

- The l.e.s. then follows from l.e.s. in rel. homotopy:

$$\begin{tikzcd}
\pi_n(F, x_0) \arrow[r, "i_*"] & \pi_n(E, x_0) \arrow[r, "j_*"] \arrow[dr, "p_*"'] & \pi_n(E, F, x_0) \arrow[r] \arrow[d, "\simeq"', "p_*"] & \pi_{n-1}(F, x_0) \arrow[r] & \dots \\
& & \pi_n(B, b_0) \arrow[ur] & &
\end{tikzcd}$$

and surjectivity of $\pi_0(F) \to \pi_0(E)$ at the end follows from
path connectedness of B and homotopy lifting property: lift path from
any $b \in B$ to b_0 to a path from any $x \in E$ to a point of F. ▲

In practice, nicest class of fibrations = <u>fiber bundles</u>

Def: <u>Fiber bundle</u> structure on E with fiber F := projection map $p: E\to B$ s.t. each point of B has a nbd U s.t. $\exists$ homeomorphism $h: p^{-1}(U)\to U\times F$ s.t. diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "h", "\sim"'] \arrow[dr, "p"'] & & U\times F \arrow[dl, "\text{proj}"] \\ & U & \end{tikzcd}$$

is commutative.

<u>Rmk:</u> hence h identifies each fiber $F_b = p^{-1}(b) \xrightarrow[\text{homeo}]{\sim} F$ for $b\in U$.

Such h is called a <u>local trivialization</u> of the bundle.

<u>Rmk:</u> the difference b/w two local trivializations h_1 over U_1, h_2 over U_2 $(U_{12} = U_1\cap U_2 \neq \emptyset)$:

$$\begin{tikzcd} U_{12}\times F \arrow[rr, bend left] \arrow[dr] & p^{-1}(U_{12}) \arrow[l, "h_1"'] \arrow[r, "h_2"] \arrow[d, "p"] & U_{12}\times F \arrow[dl] \\ & U_{12} & \end{tikzcd}$$

$$h_2\circ h_1^{-1}(b,f) = (b, \varphi_{12}(b,f))$$

ie. $\varphi_{12}: U_{12}\to \text{Homeo}(F)$ "change of trivialization".

Later we'll look at bundles with extra structure, where φ_{12} is constrained to lie in a specific subgroup of homeos; most notably <u>vector bundles</u>, with fiber = a vector space V, and we fix an equivalence class of local trivializations s.t. transition functions φ_{ij} take values in $GL(V)$. (linear automorphisms).

<u>Ex:</u>
- a fiber bundle w/ fiber a discrete space is a covering space. Conversely, a covering space over a connected base (so $|\text{fiber}|$ = constant) is a fiber bundle.
- Möbius band $= I\times[-1,1]/(0,v)\sim(1,-v)$ fiber bundle w/ fiber $[-1,1]$.
$$\downarrow$$
$$I/0\sim 1 = S^1$$
- projective spaces: complex analogue of double cover $S^n\to\mathbb{RP}^n$ $(=S^0\text{-bundle})$ is a fiber bundle $S^1\to S^{2n+1}\to \mathbb{CP}^n$.

S^{2n+1} = unit sphere in $\mathbb{C}^{n+1}$

$\mathbb{CP}^n$ = set of lines in $\mathbb{C}^{n+1}$ $= S^{2n+1}/(z_0\ldots z_n)\sim(\lambda z_0\ldots\lambda z_n)\ \forall\lambda\in S^1$

To see local triviality: over $U_i = \{[z_0\ldots z_n]/z_i\neq 0\}$, $p^{-1}(U_i)\xrightarrow{\sim} U_i\times S^1$

$$(z_0\ldots z_n)\mapsto([z_0\ldots z_n], z_i/|z_i|)$$

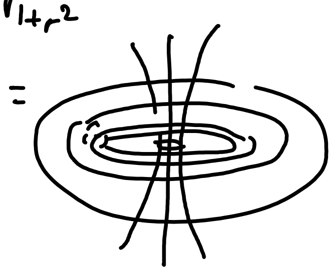
In fact, these glue together to a fiber bundle $S^1 \to S^\infty \to \mathbb{CP}^\infty$

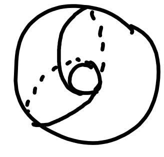
Particularly interesting: the <u>Hopf bundle</u> $S^1 \to S^3 \to S^2 (= \mathbb{CP}^1)$.

$(z_1, z_2) \mapsto z_2/z_1 \in \mathbb{C} \cup \infty$.

Rmk: preimages of circles  are tori $|z_1| = \frac{1}{\sqrt{1+r^2}}$, $|z_2| = \frac{r}{\sqrt{1+r^2}}$ $= S^1 \times S^1$

$|z_2/z_1| = r$

$S^3 = \mathbb{R}^3 \cup \infty =$ 

& on each torus, fibers = 

(linking number of fibers w/ each other is 1 !)

- over quaternions: $S^3 \to S^{4n+3} \to \mathbb{HP}^n$, eg. $S^3 \to S^7 \to S^4$

(and for octonions $S^7 \to S^{15} \to S^8$; doesn't really extend $- \mathbb{OP}^n$? $\mathbb{O}$ not assoc.!) so $(z_0 .. z_n)$ and $(z_0 .. z_n)$ not equiv relation. $\exists\ \mathbb{OP}^2$ though.)

$\mathbb{OP}^2 = S^8 \cup e^{16}$ attach by Hopf map.

just as for $\mathbb{CP}^2$ & $\mathbb{HP}^2$.

Prop: Fiber bundles have the homotopy lifting property wrt all CW-pairs (in fact, <u>all</u> spaces if base is paracompact). We don't care.

<u>Pf:</u> enough to show for disks or cubes.

Let $p: E \to B$ fiber bundle w/ fiber F,

$G: I^n \times I \to B$ a homotopy to lift, starting w/ $\tilde{g}_0$ lift of g_0 to F.

$G(x,t) = g_t(x)$

Choose open cover $\{U_\alpha\}$ of B with local trivializations $h_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F$.

Compactness of $I^n \times I \Rightarrow$ can subdivide I^n into small cubes C_i

I into small intervals $I_j = [t_j, t_{j+1}]$

st. G maps each $C_i \times I_j$ to a single U_α.

By induction on n, can assume $\tilde{g}_t$ has already been constructed over ∂C_i $\forall i$.

To extend over C_i, proceed in stages over each time interval I_j.

Thus can assume $G: I^n \times I \to U_\alpha$, and given $\tilde{G}: \underbrace{(I^n \times 0) \cup (\partial I^n \times I)}_{J} \to p^{-1}(U_\alpha) \simeq U_\alpha \times F$.

However, product case $\Rightarrow$ homotopy lifting holds! Namely:

$\tilde{g}_t(x) = (g_t(x), h_t(x))$ map to F, already given on J; extend using $I^n \times I \xrightarrow{\text{retraction}} J \xrightarrow{h} F$

Thus fiber bundles give l.e.s in homotopy.

<u>Ex</u>: • Covering space $p: E \to B$, E & B path-connected (F discrete):

$$\Rightarrow \pi_n(E) \underset{\sim}{\xrightarrow{p_*}} \pi_n(B) \quad \forall n \geq 2 \qquad (\pi_n(F), \pi_{n-1}(F) = 0).$$

and $\quad 0 \to \pi_1(E) \to \pi_1(B) \to \pi_0(F) \to 0 \quad$ (already known!)

• $S^1 \to S^\infty \to \mathbb{C}P^\infty$ gives $\pi_n(\mathbb{C}P^\infty) \simeq \pi_{n-1}(S^1)$ so $\mathbb{C}P^\infty$ is $K(\mathbb{Z}, 2)$.

• $S^1 \to S^3 \to S^2$ gives $\pi_2(S^2) \simeq \pi_1(S^1) = \mathbb{Z}$

and for $n \geq 3$, $\pi_n(S^3) \underset{\simeq}{\xrightarrow{p_*}} \pi_n(S^2)$, in particular $\pi_3(S^2) = \mathbb{Z}$.

gen^d by Hopf map.

(<u>Cor</u>: S^2 and $S^3 \times \mathbb{C}P^\infty$ are 1-connected & have same homotopy groups, but of course not homotopy equiv^t).

<u>Ex</u>: <u>Whitehead products</u>. Let's compute $\pi_3(S^2 \vee S^2)$. (more general in book)

Consider $S^2 \vee S^2 \overset{i}{\hookrightarrow} S^2 \times S^2$ $(= (S^2 \vee S^2) \cup$ 4-cell$)$.

Observe $\forall n$, $\pi_n(S^2 \vee S^2) \xrightarrow[i_*]{} \pi_n(S^2 \times S^2) \simeq \pi_n(S^2) \oplus \pi_n(S^2)$ is surjective

maps $S^n \to S^2 \times \{x_0\} \cap S^2 \vee S^2$ $\qquad$ maps $S^n \to \{x_0\} \times S^2 \cap S^2 \vee S^2$

So l.e.s. for pair splits into s.e.s.

$$0 \to \pi_{n+1}(S^2 \times S^2, S^2 \vee S^2) \to \pi_n(S^2 \vee S^2) \underset{i_*}{\twoheadrightarrow} \pi_n(S^2 \times S^2) \to 0$$

For $n = 3$: $\quad 0 \to \mathbb{Z} \to \pi_3(S^2 \vee S^2) \to \mathbb{Z}^2 \to 0$

by Hurewicz, gen^d by the 4-cell.

or by excision $\pi_4(S^2 \times S^2, S^2 \vee S^2) \simeq \pi_4(S^2 \times S^2 / S^2 \vee S^2) = \pi_4(S^4) = \mathbb{Z}$.

Hence $\pi_3(S^2 \vee S^2) \simeq \mathbb{Z}^3$, generated by:
- Hopf maps $S^3 \to$ either S^2
- attaching map of 4-cell of $S^2 \times S^2$

This is a special case of the <u>Whitehead product</u> $\pi_k(X) \times \pi_\ell(X) \to \pi_{k+\ell-1}(X)$

given $f: S^k \to X$, $g: S^\ell \to X$, let $[f,g] := S^{k+\ell-1} \xrightarrow[\text{attaching map of top cell in } S^k \times S^\ell]{} S^k \vee S^\ell \xrightarrow[f \vee g]{} X$

In our case: the unexpected gen. of $\pi_3(S^2 \vee S^2)$ is $[i_1, i_2]$, $i_1, i_2 \in \pi_2(S^2 \vee S^2)$ incl. of S^2 factors.

Ex: • $S^1 \to S^{2n-1} \to \mathbb{C}P^{n-1}$ generalizes to: (case $k=1$ of)

$$U(k) \longrightarrow V_k(\mathbb{C}^n) \longrightarrow G_k(\mathbb{C}^n) \qquad \text{fiber bundle}$$

unitary gp

$V_k(\mathbb{C}^n)$: Stiefel mfld := space of k-frames ie. $(v_1, \dots, v_k)$ orthonormal vectors in $\mathbb{C}^n$ $\longmapsto$ $\mathrm{span}(v_1, \dots, v_k) \subset \mathbb{C}^n$

$G_k(\mathbb{C}^n)$: Grassmannian of k-planes in $\mathbb{C}^n$

space of unitary bases of a given k-dim^l subspace $\simeq U(k)$

Similarly over $\mathbb{R}$ with $O(k)$ fiber, $\mathbb{H}$ with $Sp(k)$ fiber.

• For $k=1$, V_1 = sphere is highly conn^d; generally: $\begin{cases} V_k(\mathbb{R}^n) \text{ is } (n-k-1)\text{-conn.} \\ V_k(\mathbb{C}^n) \quad (2n-2k) \\ V_k(\mathbb{H}^n) \quad (4n-4k+2) \end{cases}$

and $V_k\left(\begin{matrix}\mathbb{R}^\infty \\ \mathbb{C}^\infty \\ \mathbb{H}^\infty\end{matrix}\right)$ is contractible.

This is proved by induction on k, using $V_{k-1}(\mathbb{R}^{n-1}) \to V_k(\mathbb{R}^n) \xrightarrow{p} S^{n-1}$

$(e_1, \dots, e_k) \mapsto e_1$

$\left(p^{-1}(e_1) = \{(k-1)\text{-frames in } e_1^\perp \subset \mathbb{R}^{n-1}\} \right)$ & induced l.e.s.

(similarly $\mathbb{C}$, $\mathbb{H}$)

• Specializing to $k=n$, noting $V_n(\mathbb{R}^n) \simeq O(n)$:

we get a fiber bundle $O(n-1) \to O(n) \to S^{n-1}$

$(e_1 \cdots e_n) \mapsto e_1$

ie. $A \mapsto A \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

similarly $U(n-1) \to U(n) \to S^{2n-1}$

$Sp(n-1) \to Sp(n) \to S^{4n-1}$

MONDAY 2/27

Hence homotopy groups of $O(n)$, $U(n)$, $Sp(n)$ related to those of spheres!

Corollary: The inclusion $O(n-1) \hookrightarrow O(n)$ induces an isom. on π_i for $i \leq n-3$. Hence $\pi_i O(n)$ indpt of n for $n \gg 1$.

Similarly for $U(n)$, $Sp(n)$. Surprising fact: these limit groups have a simple structure!

<u>Bott periodicity thm:</u> For $n \gg i$, $\pi_i U(n) = \begin{cases} 0 & i \equiv 0 \bmod 2 \\ \mathbb{Z} & i \equiv 1 \bmod 2 \end{cases}$ 2-periodic

$$\pi_i O(n) = \begin{cases} \mathbb{Z}_2 & i \equiv 0,1 \bmod 8 \\ \mathbb{Z} & i \equiv 3,7 \bmod 8 \\ 0 & \text{else} \end{cases} \quad \text{8 periodic}$$

$$\pi_i Sp(n) = \pi_{i+4} O(n).$$

Comes from an argument showing $\underset{\text{loop space}}{\Omega U(\infty)} \underset{\text{h.e.}}{\simeq} \mathbb{Z} \times G_\infty(\mathbb{C}^\infty)$ $= \bigcup_k G_k(\mathbb{C}^\infty)$ $\begin{pmatrix} G_k(\mathbb{C}^n) \hookrightarrow G_{k+1}(\mathbb{C} \times \mathbb{C}^n) \\ V \mapsto \mathbb{C} \times V \end{pmatrix}$

$$\Rightarrow \pi_{i+1}(U(\infty)) \simeq \pi_i(\Omega U(\infty)) \simeq \pi_i(G_\infty(\mathbb{C}^\infty)) \underset{\substack{\uparrow \\ \text{les., } V_\infty(\mathbb{C}^\infty) \text{ contractible}}}{\simeq} \pi_{i-1}(U(\infty))$$

- The stabilization properties of π_i of Lie groups are remarkably simple.
  For other spaces: Freudenthal $\Rightarrow$ $\pi_i(X) \to \pi_{i+1}(SX) \to \pi_{i+2}(S^2X) \to \cdots$
  iso if X $\frac{i}{2}$-connected; iso if SX $\frac{i+1}{2}$-connected ....
  are eventually isoms.

$\rightsquigarrow$ stable homotopy groups $\pi_i^s(X) =$ limit of this.

For spheres; $\pi_i^s := \pi_i^s(S^0) = \pi_{i+n}(S^n)$ for $n > i+1$.

<u>Thm</u> (Serre): These are all finite groups for $i > 0$.

First few:

i	0	1	2	3	4	.... (gets crazy)
π_i^s	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	...

There's an interesting operation: composition $S^{i+j+k} \to S^{j+k} \to S^k$ $k \gg i, j$
induces a map $\pi_i^s \times \pi_j^s \to \pi_{i+j}^s$

These composition maps make $\bigoplus_{i \geq 0} \pi_i^s$ a graded-commutative ring:
$\alpha\beta = (-1)^{ij} \beta\alpha$

Also, since we've been discussing fiber bundles & more generally fibrations:

- **Prop:** $p: E \to B$ fibration over path-connected $B \Rightarrow$ the fibers $F_b = p^{-1}(b)$ are all homotopy eq. [vs homeomorphic for a fiber bundle]

(idea: homotopy lifting prop applied to $f: F_{b_0} \times I \to B$
$(x,t) \mapsto \gamma(t)$ path $b_0 \rightsquigarrow b_1 \in B$
& given lift $\tilde{f}_0 = \text{inclusion}: F_{b_0} \to E$
HLP gives $\tilde{f}_1 : F_{b_0} \to F_{b_1}$ ... show h.e. by considering converse map $F_{b_1} \to F_{b_0}$ & homotopy to id)

- Pullback construction:

$p: E \to B$ fibration (resp fiber bundle), $f: A \to B$
$\Rightarrow$ pullback fibration (resp. bundle) $f^*E = \{(a,x) \in A \times E \,/\, f(a) = p(x)\} \xrightarrow[\text{proj. to 1st factor}]{\pi} A$

commutative diagram
$$\begin{array}{ccc} f^*E & \to & E \\ \pi \downarrow & & \downarrow p \\ A & \xrightarrow{f} & B \end{array}$$

Easy to check: homotopy lifting prop. for $p \Rightarrow$ for π as well
local triv. for $p \Rightarrow$ for π.

Prop: $f_0, f_1 : A \to B$ homotopic, $p: E \to B$ fibration $\Rightarrow f_0^*E, f_1^*E$ are fiber-preserving homotopy equivalent.
i.e. $\exists$
$$\begin{array}{ccc} f_0^*E & \xrightarrow[\simeq]{\text{h.e.}} & f_1^*E \\ & {}_{\pi_0}\searrow \;\; \swarrow_{\pi_1} & \\ & A & \end{array}$$

(use: $F: A \times I \to B$ homotopy, then $\begin{array}{c} F^*E \\ \downarrow \\ A \times I \end{array}$ fibration, lift $f_0^*E \times I \xrightarrow[\pi_0 \times id]{} A \times I$
w/ given lift on $f_0^*E \times \{0\} = \text{inclusion}$)
to get fiber-preserving map $f_0^*E \to f_1^*E$

[similar statement for fiber bundles ... f_0^*E, f_1^*E fiber-preserving homeomorphic].

Corollary: A fibration over a contractible base is fiber-preserving homotopy eq. to a product fibration $p: F \times B \to B$.

(apply prop. to $id^*E = E$ vs. $r^*E = F_{b_0} \times B$ where $r: B \to \{b_0\}$).